THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H/I/J University Mathematics 2017-2018
Assignment 4
Due Date: 5 Mar 2018 (Mon)

1. Find $\frac{dy}{dx}$ if

   (a) $y = \log_{10}(2^x + 5x)$

   (b) $y = \sin x \ln x$

   (c) $y = x \csc x$

   (d) $y = \frac{\sin x}{x}$

   (e) $y = \sqrt{x^3 e^x + 1}$

   (f) $y = e^{\cot x}$

   (g) $y = \sqrt[3]{\frac{e^{7x}(x^2+1)^3}{(x+1)^5}}$

   (h) $y = x^{\cos x}$

   (i) $y = \tan^{-1} x$

2. Let $\mathcal{C}$ be a curve defined by the equation $x^3 + 2xy + y^2 = 1$.

   Show that $(0, 1)$ is a point lying on $\mathcal{C}$ and find the equation of tangent line at that point.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Show that if f is an odd function, then its derivative f' is an even function.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 + x + x^2 & \text{if} \quad x \geq 0, \\ 1 + x & \text{if} \quad x < 0. \end{cases}$$

   (a) Show that $f(x)$ is differentiable at $x = 0$.

   (b) Write down the function $f'(x)$ explicity.

   (c) Is $f'(x)$ differentiable at $x = 0$? Why?

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

   (a) Show that $f(x)$ is differentiable at $x = 0$.

   (b) Write down the function $f'(x)$ explicity.

   (c) Is $f'(x)$ differentiable at $x = 0$? Why?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant function satisfying

   - $f(x + y) = f(x) + f(y) + f(x)f(y)$ for all $x, y \in \mathbb{R}$
   - $\lim_{h \to 0} \frac{f(h)}{h} = a$, where a is a real number

   (a) Show that

      (i) $f(0)(1 + f(x)) = 0$ for all $x \in \mathbb{R}$;

      (ii) $f(0) = 0$;

(iii) $f(x) \neq -1$ for all $x \in \mathbb{R}$.

(Hint: Suppose that $f(y_0) = -1$ for some $y_0 \in \mathbb{R}$, consider $f(x) = f((x - y_0) + y_0)$ to obtain contradiction.)

(b) By considering $f(x) = f(\frac{x}{2} + \frac{x}{2})$, show that $f(x) > -1$ for all $x \in \mathbb{R}$.

(c) Prove that f is a differentiable function and $f'(x) = a(1 + f(x))$ for all $x \in \mathbb{R}$.

Furthermore, show that $a \neq 0$.

(d) By considering the derivative of the function $\ln(1 + f(x))$, prove that $f(x) = e^{ax} - 1$.

7. By using the mean value theorem, show that

(a) $\frac{x}{1+x} < \ln(1 + x) < x$ for $x > 0$;

(b) $ny^{n-1}(x - y) < x^n - y^n < nx^{n-1}(x - y)$ for $n > 1$ and $0 < y < x$.

8. Show that $\ln(1 + x) - \frac{2x}{2 + x} < \frac{x^3}{12}$ for $x > 0$.

9. If $f(x) = \frac{(x + n + 1)^{n+1}}{(x + n)^n}$ where $x > 0$ and n is a positive integer.

Show that $f(x)$ is strictly increasing on $(0, +\infty)$. Hence show that

$$(1 + \frac{1}{n})^n < (1 + \frac{1}{n + 1})^{n+1}.$$

10. (a) Let $h(x) = \frac{x}{\ln x}$ for $x > 1$. Show that $h(x) \geq e$ for all $x > 1$.

(b) Let $f(x) = \frac{x^b}{b^x}$ for $b, x > 1$. Show that $f(x)$ is strictly increasing on $(1, \frac{b}{\ln b})$ and strictly decreasing on $(\frac{b}{\ln b}, +\infty)$.

(c) Using (b) to show that if $1 < a < b < e$, then $a^b < b^a$.